

CESÀRO SUMMABILITY OF FOURIER ORTHOGONAL EXPANSIONS ON THE CYLINDER

JEREMY WADE

ABSTRACT. A result concerning the Cesàro summability of the Fourier orthogonal expansion of a function on the cylinder, where the orthogonal basis consists of orthogonal polynomials, in the L^p norms is presented. An upper bound for critical index δ is obtained.

1. INTRODUCTION

Cesàro summability of an infinite series is a classical topic which still receives a considerable amount of attention. In particular, research into the Cesàro summability of Fourier expansions in a multivariate setting is quite active. For example, the Cesàro summability of a Fourier orthogonal expansion in orthogonal polynomials on the parabolic biangle is studied in [1], and the unit ball and sphere are studied in [2]. Investigations into approximation on the cylinder have also received some attention recently. In [3], approximation using Fekete and Leja points is studied, while in [9], an approximation technique relating orthogonal polynomials and the Radon transform on parallel disks was investigated. In this paper, we present a result on the convergence of the Cesàro means of a Fourier orthogonal expansion in orthogonal polynomials of a function defined on the cylinder. This result relies on results proven by Xu and Li in [6], which concerns the Cesàro summability of functions on the hypercube.

2. BACKGROUND

We denote the d -dimensional unit ball by B^d and the m -dimensional hypercube $[-1, 1]^m$ by I^m . We first present the orthogonal polynomial basis used to obtain the Fourier orthogonal decomposition on the cylinder. We will use the notation n_d to denote the dimension of the space of orthogonal polynomials of degree n on either B^d or I^d , which is known to satisfy

$$n_d = \binom{n+d-1}{n}.$$

On the hypercube I^m , the product Jacobi polynomials are used to form a basis. The univariate Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ form an orthogonal polynomial basis on $[-1, 1]$ with respect to the weight function $w^{(\alpha, \beta)}(x) = (1-x)^\alpha(1+x)^\beta$, with $\alpha, \beta > -1$; that is,

$$(2.1) \quad \int_{-1}^1 P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) w^{(\alpha, \beta)}(x) dx = \delta_{m,n} c_{n, \alpha, \beta},$$

Date: November 4, 2011.

2000 Mathematics Subject Classification. 41A35, 41A63, 42A24.

Key words and phrases. Cesàro Summability, Cylinder, Multidimensional Approximation.

where

$$c_{n,\alpha,\beta} = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)},$$

see [8] for more details. The orthonormal Jacobi polynomials are obtained by scaling the Jacobi polynomials so that the right side of (2.1) is $\delta_{m,n}$; we denote the resulting orthonormal polynomials by $p_n^{(\alpha,\beta)}(x)$. Next, an orthonormal product Jacobi polynomial on I^m is defined by

$$(2.2) \quad P_\gamma^{(\alpha,\beta)}(\mathbf{x}) = p_{\gamma_1}^{(\alpha_1,\beta_1)}(x_1) p_{\gamma_2}^{(\alpha_2,\beta_2)}(x_2) \cdots p_{\gamma_m}^{(\alpha_m,\beta_m)}(x_m)$$

where we let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, and similarly define β , γ , and \mathbf{x} . We will consider the degree of $P_\gamma^{(\alpha,\beta)}(\mathbf{x})$ to be the total degree, $|\gamma| := \gamma_1 + \gamma_2 + \cdots + \gamma_m$. With the weight function $w^{(\alpha,\beta)}(\mathbf{x}) = \prod_{i=1}^m w^{(\alpha_i,\beta_i)}(x_i)$, an orthonormal basis for the space of polynomials of degree n on I^m is obtained by taking the set of all polynomials of the form in (2.2), with $|\gamma| = n$, where orthonormality is in the sense that

$$(2.3) \quad \int_{I^m} P_\gamma^{(\alpha,\beta)}(\mathbf{x}) P_\eta^{(\alpha,\beta)}(\mathbf{x}) w^{(\alpha,\beta)}(\mathbf{x}) d\mathbf{x} = \delta_{\gamma,\eta}.$$

It will be convenient to follow the notation developed in Chapter 2 of [4] and list the orthonormal basis of degree n polynomials on I^m in column vector form. We define

$$\mathbb{P}_n^{(\alpha,\beta)}(\mathbf{x}) = \begin{bmatrix} P_{\gamma^1}^{(\alpha,\beta)}(\mathbf{x}) \\ P_{\gamma^2}^{(\alpha,\beta)}(\mathbf{x}) \\ \vdots \\ P_{\gamma^{n_m}}^{(\alpha,\beta)}(\mathbf{x}) \end{bmatrix},$$

where $\gamma^1, \gamma^2, \dots, \gamma^{n_m}$ are multi-indices with $|\gamma| = n$.

On the d -dimensional unit ball B^d , we consider the weight function $w_\mu(\mathbf{y}) = (1 - \|\mathbf{y}\|^2)^{\mu-1/2}$, with $\mu \geq 0$, and denote a basis of orthonormal polynomials of degree $n = |\alpha|$ on B^d by $s_{\alpha^1}^\mu(\mathbf{y}), s_{\alpha^2}^\mu(\mathbf{y}), \dots, s_{\alpha^{n_d}}^\mu(\mathbf{y})$, where orthonormality is in a similar manner as (2.3). Several examples of specific orthonormal bases exist, see [4, p. 38]. If $d = 1$ and $\mu = 0$, these polynomials correspond to the Chebyshev polynomials of the first kind, while if $d = 2$ and $\mu = 1/2$, an orthonormal basis is given by the polynomials

$$2^{n/2-j+1} p_j^{(0,n-2j)}(2\|\mathbf{y}\|^2 - 1) S_{\beta,n-2j}(\mathbf{y}),$$

where $0 \leq j \leq n$ and $S_{\beta,n-2j}(\mathbf{y})$ is a spherical harmonic on S^1 , defined by

$$S_{1,n}(\theta) = \frac{1}{\sqrt{\pi}} \sin(n(\pi/2 - \theta)), \quad S_{2,n}(\theta) = \frac{1}{\sqrt{\pi}} \cos(n(\pi/2 - \theta)),$$

with $x = \cos \theta$ and $y = \sin \theta$; see [7] for more information on spherical harmonics. One important property of orthogonal polynomials on B^d with respect to w^μ is the

compact formula introduced in [10],

$$(2.4) \quad [\mathbb{S}_n^\mu(\mathbf{y})]^T \mathbb{S}_n^\mu(\mathbf{y}') = \frac{n + \mu + \frac{d-1}{2}}{\mu + \frac{d-1}{2}} \frac{(\Gamma(\mu))^2}{2^{2\mu-1} \Gamma(2\mu)} \\ \times \int_{-1}^1 C_n^{(\mu + \frac{d-1}{2})}(\mathbf{x} \cdot \mathbf{y} + t \sqrt{1 - |\mathbf{x}|^2} \sqrt{1 - |\mathbf{y}|^2}) (1 - t^2)^{\mu-1} dt,$$

for $\mu > 0$, and

$$(2.5) \quad [\mathbb{S}_n^0(\mathbf{y})]^T \mathbb{S}_n^0(\mathbf{y}') = \frac{n + \frac{d-1}{2}}{\frac{d-1}{2}} \left[C_n^{(\frac{d-1}{2})}(\mathbf{x} \cdot \mathbf{y} + \sqrt{1 - |\mathbf{x}|^2} \sqrt{1 - |\mathbf{y}|^2}) \right. \\ \left. + C_n^{(\frac{d-1}{2})}(\mathbf{x} \cdot \mathbf{y} - \sqrt{1 - |\mathbf{x}|^2} \sqrt{1 - |\mathbf{y}|^2}) \right]$$

for $\mu = 0$.

Following the column vector notation used on I^m , we will use the notation

$$\mathbb{S}_n^\mu(\mathbf{y}) = \begin{bmatrix} s_{\alpha^1}^\mu(\mathbf{y}) \\ s_{\alpha^2}^\mu(\mathbf{y}) \\ \vdots \\ s_{\alpha^{n_d}}^\mu(\mathbf{y}) \end{bmatrix}$$

to denote the column vector whose elements form an orthonormal basis of polynomials of degree n on B^d with respect to w_μ .

The reproducing kernel of degree n with respect to a weight function w on some set X of positive Borel measure, $K_n(\mathbf{x}, \mathbf{x}')$, is a function satisfying

$$\int_X f(\mathbf{x}) K_n(\mathbf{x}, \mathbf{x}') w(\mathbf{x}) d\mathbf{x} = f(\mathbf{x}')$$

when f is a polynomial of degree less than or equal to n . Given a basis of orthonormal polynomials with respect to w , $\{P_\alpha(\mathbf{x})\}_{\alpha \in A}$, for the space of polynomials of degree less than or equal to n , the reproducing kernel has the form

$$K_n(\mathbf{x}, \mathbf{x}') = \sum_{\alpha \in A} P_\alpha(\mathbf{x}) P_\alpha(\mathbf{x}').$$

The n 'th partial sum of the Fourier orthogonal expansion of an integrable function f on X with respect to w , $S(w; f)$, is defined by

$$S_n(w; f)(\mathbf{x}') = \int_X f(\mathbf{x}) K_n(\mathbf{x}, \mathbf{x}') w(\mathbf{x}) d\mathbf{x}.$$

With the column vector notation above, the reproducing kernel on the hypercube with respect to $w^{(\alpha, \beta)}$ can be written as

$$K_n(\mathbf{x}, \mathbf{x}') = \sum_{k=0}^n \left[\mathbb{P}_k^{(\alpha, \beta)}(\mathbf{x}) \right]^T \mathbb{P}_k^{(\alpha, \beta)}(\mathbf{x}'),$$

and a similar formula holds for the reproducing kernel on the ball with respect to w_μ . We may then write the reproducing kernel on $B^d \times I^m$ with respect to the

weight function $w(\alpha, \beta, \mu; \mathbf{x}, \mathbf{y}) := w^{(\alpha, \beta)}(\mathbf{x})w_\mu(\mathbf{y})$, $K_n(\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}')$, as

$$K_n(\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}') = \sum_{k=0}^n \sum_{j=0}^k \left[\mathbb{P}_j^{(\alpha, \beta)}(\mathbf{x}) \right]^T \mathbb{P}_j^{(\alpha, \beta)}(\mathbf{x}') \left[\mathbb{S}_{k-j}^\mu(\mathbf{y}) \right]^T \mathbb{S}_{k-j}^\mu(\mathbf{y}').$$

We define the n 'th partial sum of the Fourier orthogonal expansion of an integrable function f on $B^d \times I^m$ to be

$$S_n(\mu, \alpha, \beta; f)(\mathbf{x}', \mathbf{y}') = \int_{B^d} \int_{I^m} K_n(\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}') f(\mathbf{x}, \mathbf{y}) w(\alpha, \beta, \mu; \mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}.$$

Given a series $\sum s_n$, the Cesàro means, or (C, δ) means, of the series is defined to be

$$\sum_{j=0}^n c_{n,j}^\delta \sum_{k=0}^j s_k,$$

where $c_{n,j}^\delta = \frac{(-n)_j}{(-n-\delta)_j}$, and $(n)_j = \prod_{k=1}^j (n+k-1)$. If we define

$$K_n^\delta(\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}') = \sum_{k=0}^n c_{n,k}^\delta \sum_{j=0}^k \left[\mathbb{P}_j^{(\alpha, \beta)}(\mathbf{x}) \right]^T \mathbb{P}_j^{(\alpha, \beta)}(\mathbf{x}') \left[\mathbb{S}_{k-j}^\mu(\mathbf{y}) \right]^T \mathbb{S}_{k-j}^\mu(\mathbf{y}'),$$

then the Cesàro means of order δ , or the (C, δ) means, of the Fourier orthogonal expansion of f are defined by

$$S_n^\delta(\mu, \alpha, \beta; f)(\mathbf{x}', \mathbf{y}') = \int_{B^d} \int_{I^m} K_n^\delta(\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}') f(\mathbf{x}, \mathbf{y}) w(\alpha, \beta, \mu; \mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}.$$

We will be investigating the value of δ for which this series converges in the space $L^p(B^d \times I^m; w(\alpha, \beta, \mu; \mathbf{x}, \mathbf{y}))$ - that is, for what δ is

$$\lim_{n \rightarrow \infty} \int_{B^d \times I^m} |S_n^\delta(\alpha, \beta, \mu; f)(\mathbf{x}, \mathbf{y}) - f(\mathbf{x}, \mathbf{y})|^p w(\alpha, \beta, \mu; \mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} = 0.$$

3. MAIN THEOREM

We now present our main result for this paper.

Theorem 3.1. *Let f be a continuous function on $B^d \times I^m$, and suppose that $\mu \geq 0$, $\alpha_i > -1$, $\beta_i > -1$, and $\alpha_i + \beta_i \geq -1$ for $1 \leq i \leq m$. The Cesàro means of the Fourier orthogonal expansion of f with respect to $w(\alpha, \beta, \mu; \mathbf{x}, \mathbf{y})$ converge in $L^p(B^d \times I^m; w(\alpha, \beta, \mu; \mathbf{x}, \mathbf{y}))$, with $1 \leq p < \infty$, and $C(B^d \times I^m)$, to f if*

$$(3.1) \quad \delta > \sum_{i=1}^m \max\{\alpha_i, \beta_i\} + \mu + \frac{d+m-1}{2} + \max \left\{ 0, -\sum_{i=1}^m \min\{\alpha_i, \beta_i\} - \mu - \frac{d+m+1}{2} \right\}.$$

Proof. Our proof will ultimately reduce the question of convergence on the cylinder to that of the hypercube. We will show that the integral

$$\int_{I^m} \int_{B^d} |K_n^\delta(\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}')| w(\alpha, \beta, \mu; \mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}$$

is uniformly bounded by some constant M which is independent of n , \mathbf{x}' , and \mathbf{y}' , which will then imply summability, by results known on the hypercube. Throughout

our proof, c will denote a positive constant that may change values from line to line.

Following Lemma 2.2 in [6], we first show that it is enough to consider $\mathbf{x}' = \mathbf{e} := (1, 1, \dots, 1)$ in the hypercube.

Lemma 3.2. *In order to prove the convergence of (C, δ) means of the orthogonal expansion, it suffices to prove*

$$(3.2) \quad \int_{I^m} \int_{B^d} |K_n^\delta(\mathbf{x}, \mathbf{e}, \mathbf{y}, \mathbf{y}')| w(\alpha, \beta, \mu; \mathbf{x}, \mathbf{y}) d\mathbf{y} d\mathbf{x} \leq M$$

for M independent of n and \mathbf{y}' .

Proof. The proof of the lemma follows from the following theorem, which appears in [5, p. 262].

Theorem 3.3. *Let $\alpha, \beta > -1$ and $\alpha \geq \beta$. An integral representation of the form*

$$P_n^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(y) = \int_{-1}^1 P_n^{(\alpha, \beta)}(1) P_n^{(\alpha, \beta)}(z) K^{(\alpha, \beta)}(x, y, z) w^{(\alpha, \beta)}(z) dz$$

exists, with the function K satisfying

$$\int_{-1}^1 |K^{(\alpha, \beta)}(x, y, z)| w^{(\alpha, \beta)}(z) dz \leq M$$

for $-1 < x, y < 1$.

This result is for univariate Jacobi polynomials, but easily extends to the product Jacobi polynomials as

$$P_n^{(\alpha, \beta)}(\mathbf{x}) P_n^{(\alpha, \beta)}(\mathbf{y}) = \int_{I^m} P_n^{(\alpha, \beta)}(\mathbf{e}) P_n^{(\alpha, \beta)}(\mathbf{z}) \mathbf{K}^{(\alpha, \beta)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) w^{(\alpha, \beta)}(\mathbf{z}) d\mathbf{z},$$

where $\mathbf{x}, \mathbf{y} \in I^m$, and

$$\mathbf{K}^{(\alpha, \beta)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \prod_{i=1}^m K^{(\alpha_i, \beta_i)}(x_i, y_i, z_i),$$

and $\mathbf{K}^{(\alpha, \beta)}(\cdot, \cdot, \cdot)$ satisfies

$$\int_{I^m} |\mathbf{K}^{(\alpha, \beta)}(\mathbf{x}, \mathbf{y}, \mathbf{z})| w^{(\alpha, \beta)}(\mathbf{z}) d\mathbf{z} \leq M,$$

where M is a constant given by the product of the constants in the univariate case. We obtain

$$\begin{aligned} A_n^\delta &:= \int_{I^m} \int_{B^d} |K_n^\delta(\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}')| w(\alpha, \beta, \mu; \mathbf{x}, \mathbf{y}) d\mathbf{y} d\mathbf{x} \\ &= \int_{I^m} \int_{B^d} \left| \sum_{j=0}^n c_{n,j}^\delta \sum_{k=0}^j [\mathbb{P}_{j-k}^{(\alpha, \beta)}(\mathbf{x})]^T \mathbb{P}_{j-k}^{(\alpha, \beta)}(\mathbf{x}') [\mathbb{S}_k^\mu(\mathbf{y})]^T \mathbb{S}_k^\mu(\mathbf{y}') \right| \\ &\quad \times w(\alpha, \beta, \mu; \mathbf{x}, \mathbf{y}) d\mathbf{y} d\mathbf{x} \\ &\leq \int_{I^m} \int_{B^d} \int_{I^m} \left| \sum_{j=0}^n c_{n,j}^\delta \sum_{k=0}^j [\mathbb{P}_{j-k}^{(\alpha, \beta)}(\mathbf{e})]^T \mathbb{P}_{j-k}^{(\alpha, \beta)}(\mathbf{z}) [\mathbb{S}_k^\mu(\mathbf{y})]^T \mathbb{S}_k^\mu(\mathbf{y}') \right| \\ &\quad \times |\mathbf{K}(\mathbf{x}, \mathbf{x}', \mathbf{z})| w^{(\alpha, \beta)}(\mathbf{z}) d\mathbf{z} w(\alpha, \beta, \mu; \mathbf{x}, \mathbf{y}) d\mathbf{y} d\mathbf{x}. \end{aligned}$$

Applying Fubini's theorem gives

$$\begin{aligned} A_n^\delta &\leq \int_{B^d} \int_{I^m} \left| \sum_{j=0}^n c_{n,j}^\delta \sum_{k=0}^j \left[\mathbb{P}_{j-k}^{(\alpha,\beta)}(\mathbf{e}) \right]^T \mathbb{P}_{j-k}^{(\alpha,\beta)}(\mathbf{z}) [\mathbb{S}_k^\mu(\mathbf{y})]^T \mathbb{S}_k^\mu(\mathbf{y}') \right| \\ &\quad \times \int_{I^m} |\mathbf{K}(\mathbf{x}, \mathbf{x}', \mathbf{z})| w^{(\alpha,\beta)}(\mathbf{x}) d\mathbf{x} w(\alpha, \beta, \mu; \mathbf{z}, \mathbf{y}) d\mathbf{z} d\mathbf{y} \\ &\leq M \int_{I^m} \int_{B^d} |K_n^\delta(\mathbf{e}, \mathbf{z}, \mathbf{y}, \mathbf{y}')| w(\alpha, \beta, \mu; \mathbf{z}, \mathbf{y}) d\mathbf{y} d\mathbf{z}. \end{aligned}$$

Replacing \mathbf{z} with \mathbf{x} and switching the places of \mathbf{e} and \mathbf{x} proves the lemma. \square

Our next lemma reduces the integral over B^d to an integral over $[-1, 1]$ of a Gegenbauer polynomial. The idea for this lemma comes from the proof of Theorem 5.3 in [10]. We define

$$\mathcal{G}_\mu^{(\alpha,\beta)}(\mathbf{y}') := \int_{B^d} |K_n^\delta(\mathbf{x}, \mathbf{e}, \mathbf{y}, \mathbf{y}')| w_\mu(\mathbf{y}) d\mathbf{y}$$

and

$$F_{n,\mu}^\delta(\cdot) := \sum_{j=0}^n c_{j,n}^\delta \sum_{k=0}^j \frac{k + \mu + \frac{d-1}{2}}{\mu + \frac{d-1}{2}} C_k^{(\mu + \frac{d-1}{2})}(\cdot) \left[\mathbb{P}_{j-k}^{(\alpha,\beta)}(\mathbf{x}) \right]^T \mathbb{P}_{j-k}^{(\alpha,\beta)}(\mathbf{e}).$$

Lemma 3.4. For $\mu \geq 0$,

$$(3.3) \quad \mathcal{G}_\mu^{(\alpha,\beta)}(\mathbf{y}') \leq c \int_{-1}^1 |F_n^\delta(u)| (1-u^2)^{\frac{d-2}{2}+\mu} du,$$

Proof. We first consider the case $\mu > 0$. Substitute (2.4) into (3.2) to obtain

$$(3.4) \quad \mathcal{G}_\mu^{(\alpha,\beta)}(\mathbf{y}') = \int_{B^d} \left| \int_{-1}^1 F_n^\delta \left(\langle \mathbf{y}, \mathbf{y}' \rangle + \sqrt{1-|\mathbf{y}|^2} \sqrt{1-|\mathbf{y}'|^2} t \right) (1-t^2)^{\mu-1} dt \right| \\ \times w_\mu(\mathbf{y}) d\mathbf{y}.$$

Applying the change of variable $\mathbf{y} = r\eta$, where $\eta \in S^{d-1}$, $0 \leq r \leq 1$, gives

$$\mathcal{G}_\mu^{(\alpha,\beta)}(\mathbf{y}') = \int_0^1 r^{d-1} \int_{S^{d-1}} \left| \int_{-1}^1 F_n^\delta \left(r \langle \eta, \mathbf{y}' \rangle + \sqrt{1-|\mathbf{y}'|^2} \sqrt{1-r^2} t \right) (1-t^2)^{\mu-1} dt \right| \\ (1-r^2)^{\mu-1/2} d\omega(\eta) dr,$$

where $d\omega$ is the surface measure on S^{d-1} . Now let A be the rotation matrix satisfying $A(\mathbf{y}') = (0, 0, \dots, 0, |\mathbf{y}'|)$, and apply the change of basis $\eta \mapsto A^T \eta$ to obtain

$$\mathcal{G}_\mu^{(\alpha,\beta)}(\mathbf{y}') = \int_0^1 r^{d-1} \int_{S^{d-1}} \left| \int_{-1}^1 F_n^\delta \left(r \eta_d |\mathbf{y}'| + \sqrt{1-|\mathbf{y}'|^2} \sqrt{1-r^2} t \right) (1-t^2)^{\mu-1} dt \right| \\ \times (1-r^2)^{\mu-1/2} d\omega(\eta) dr$$

where $\eta = (\eta_1, \dots, \eta_d)$. If we let $\eta_d = s$, then $\eta = (\sqrt{1-s^2}\gamma, s)$ for some $\gamma \in S^{d-2}$, and changing variables gives

$$(3.5) \quad \mathcal{G}_\mu^{(\alpha, \beta)}(\mathbf{y}') = \omega_{d-2} \int_0^1 r^{d-1} \int_{-1}^1 \left| \int_{-1}^1 F_n^\delta \left(rs|\mathbf{y}'| + \sqrt{1-|\mathbf{y}'|^2} \sqrt{1-r^2} t \right) (1-t^2)^{\mu-1} dt \right| \times (1-r^2)^{\mu-1/2} (1-s^2)^{\frac{d-3}{2}} ds dr$$

where ω_{d-2} is the surface area of S^{d-2} . Let $s \mapsto p/r$ so $ds = dp/r$ and move the absolute value inside the innermost integral to obtain

$$\mathcal{G}_\mu^{(\alpha, \beta)}(\mathbf{y}') \leq \omega_{d-2} \int_0^1 \int_{-r}^r \int_{-1}^1 \left| F_n^\delta \left(p|\mathbf{y}'| + \sqrt{1-|\mathbf{y}'|^2} \sqrt{1-r^2} t \right) (1-t^2)^{\mu-1} \right| dt \times (1-r^2)^{\mu-1/2} r(r^2-p^2)^{\frac{d-3}{2}} dp dr.$$

Switching the order of integration of r and p and applying the change of variable $q \mapsto \sqrt{1-r^2}t$, $dq = \sqrt{1-r^2} dt$ gives

$$\mathcal{G}_\mu^{(\alpha, \beta)}(\mathbf{y}') \leq \omega_{d-2} \int_{-1}^1 \int_{|p|}^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} \left| F_n^\delta \left(p|\mathbf{y}'| + \sqrt{1-|\mathbf{y}'|^2} q \right) \right| \times (1-r^2-q^2)^{\mu-1} dq r(r^2-p^2)^{\frac{d-3}{2}} dr dp.$$

Switching the order of integration of q and r gives

$$\mathcal{G}_\mu^{(\alpha, \beta)}(\mathbf{y}') \leq \omega_{d-2} \int_{-1}^1 \int_{-\sqrt{1-|p|^2}}^{\sqrt{1-|p|^2}} \left| F_n^\delta \left(p|\mathbf{y}'| + \sqrt{1-|\mathbf{y}'|^2} q \right) \right| \times \int_{|p|}^{\sqrt{1-q^2}} (1-r^2-q^2)^{\mu-1} r(r^2-p^2)^{\frac{d-3}{2}} dr dq dp.$$

Applying the change of variable $r^2 = u(1-q^2-p^2) + p^2$ shows the inner integral is $\frac{1}{2}(1-q^2-p^2)^{\mu+\frac{d-3}{2}} B(\mu, \frac{d-1}{2})$, where $B(x, y)$ is the beta function. Hence, we have the inequality

$$\mathcal{G}_\mu^{(\alpha, \beta)}(\mathbf{y}') \leq \frac{\omega_{d-2} B(\mu, \frac{d-1}{2})}{2} \int_{-1}^1 \int_{-\sqrt{1-|p|^2}}^{\sqrt{1-|p|^2}} \left| F_n^\delta \left(p|\mathbf{y}'| + \sqrt{1-|\mathbf{y}'|^2} q \right) \right| \times (1-q^2-p^2)^{\frac{d-3}{2}+\mu} dq dp.$$

Next, we apply the change of variable $q \mapsto \sqrt{1-p^2}s$ to obtain

$$(3.6) \quad \mathcal{G}_\mu^{(\alpha, \beta)}(\mathbf{y}') \leq c \int_{-1}^1 \int_{-1}^1 \left| F_n^\delta \left(p|\mathbf{y}'| + \sqrt{1-|\mathbf{y}'|^2} \sqrt{1-p^2} s \right) \right| \times (1-p^2)^{\frac{d-2}{2}+\mu} (1-s^2)^{\frac{d-3}{2}+\mu} ds dp.$$

Changing variables once again, we let $u = p|\mathbf{y}'| + \sqrt{1-|\mathbf{y}'|^2}\sqrt{1-p^2}s$ to obtain

$$(3.7) \quad \mathcal{G}_\mu^{(\alpha,\beta)}(\mathbf{y}') \leq c \int_{-1}^1 \int_{p|\mathbf{y}'|-\sqrt{1-|\mathbf{y}'|^2}\sqrt{1-p^2}}^{p|\mathbf{y}'|+\sqrt{1-|\mathbf{y}'|^2}\sqrt{1-p^2}} |F_n^\delta(u)| D_{\frac{d-2}{2}+\mu}(|\mathbf{y}'|, p, u) \\ \times (1-p^2)^{\frac{d-2}{2}+\mu} (1-u^2)^{\frac{d-2}{2}+\mu} du dp,$$

where the function $D_\lambda(v, p, u)$, introduced in [10], is defined by

$$D_\lambda(v, p, u) = \frac{(1-v^2-p^2-u^2+2upv)^{\lambda-1/2}}{[(1-v^2)(1-u^2)(1-p^2)]^\lambda}$$

for $1-v^2-p^2-u^2+2upv \geq 0$ and 0 otherwise. It is readily verified that

$$\int_{-1}^1 D_\lambda(u, v, p) (1-p^2)^\lambda dp = 2^{2\lambda} B(\lambda+1/2, \lambda+1/2).$$

Hence switching the order of integration in (3.7), we have

$$(3.8) \quad \mathcal{G}_\mu^{(\alpha,\beta)}(\mathbf{y}') \leq c \int_{-1}^1 |F_n^\delta(u)| \int_{-1}^1 D_{\frac{d-2}{2}+\mu}(|\mathbf{y}'|, u, p) (1-p^2)^{\frac{d-2}{2}+\mu} dp \\ \times (1-u^2)^{\frac{d-2}{2}+\mu} du \\ \leq c \int_{-1}^1 |F_n^\delta(u)| (1-u^2)^{\frac{d-2}{2}+\mu} du.$$

This proves the lemma for $\mu > 0$.

Turning our attention now to the case when $\mu = 0$, we substitute (2.5) into the left side of (3.2) and ignore the integral over I^m as before to obtain

$$\mathcal{G}_0^{(\alpha,\beta)}(\mathbf{y}') := \int_{B^d} \left| F_n^\delta \left(\langle \mathbf{y}, \mathbf{y}' \rangle + \sqrt{1-|\mathbf{y}|^2} \sqrt{1-|\mathbf{y}'|^2} \right) \right. \\ \left. + F_n^\delta \left(\langle \mathbf{y}, \mathbf{y}' \rangle \sqrt{1-|\mathbf{y}|^2} \sqrt{1-|\mathbf{y}'|^2} \right) \right| w_0(\mathbf{y}) d\mathbf{y}.$$

We perform the same change of variables from the case when $\mu > 0$ to obtain the equivalent of (3.5),

$$\mathcal{G}_0^{(\alpha,\beta)}(\mathbf{y}') = \omega_{d-2} \int_0^1 r^{d-1} \int_{-1}^1 \left| F_n^\delta \left(rs|\mathbf{y}'| + \sqrt{1-|\mathbf{y}'|^2} \sqrt{1-r^2} \right) \right. \\ \left. + F_n^\delta \left(rs|\mathbf{y}'| - \sqrt{1-|\mathbf{y}'|^2} \sqrt{1-r^2} \right) \right| (1-r^2)^{-1/2} (1-s^2)^{\frac{d-3}{2}} ds dr.$$

Now we substitute $p = \sqrt{1-r^2}$ and let $v = \sqrt{1-|\mathbf{y}'|^2}$ to obtain

$$(3.9) \quad \mathcal{G}_0^{(\alpha,\beta)}(\mathbf{y}') = \omega_{d-2} \int_{-1}^1 \int_0^1 \left| F_n^\delta \left(\sqrt{1-p^2} \sqrt{1-v^2} s + pv \right) \right. \\ \left. + F_n^\delta \left(\sqrt{1-p^2} \sqrt{1-v^2} s - pv \right) \right| (1-p^2)^{\frac{d-2}{2}} (1-s^2)^{\frac{d-3}{2}} dp ds \\ = \omega_{d-2} \int_{-1}^1 \int_{-1}^1 \left| F_n^\delta \left(\sqrt{1-p^2} \sqrt{1-v^2} s + pv \right) \right| \\ \times (1-p^2)^{\frac{d-2}{2}} (1-s^2)^{\frac{d-3}{2}} dp ds.$$

The right side of (3.9) is the right side of (3.6), with v in place of $|\mathbf{y}'|$. Following the same steps of the proof for $\mu > 0$, we obtain the equivalent of (3.8),

$$\mathcal{G}_0^{(\alpha, \beta)}(y') \leq c \int_{-1}^1 |F_n^\delta(u)| (1-u^2)^{\frac{d-2}{2}} du,$$

which proves the case for $\mu = 0$. \square

To finish the proof of Theorem 3.1, we substitute (3.3) into (3.2) to obtain

$$\begin{aligned} & \int_{I^m} \int_{B^d} |K_n^\delta(\mathbf{x}, \mathbf{e}, \mathbf{y}, \mathbf{y}')| w(\alpha, \beta, \mu; \mathbf{x}, \mathbf{y}) d\mathbf{y} d\mathbf{x} \\ & \leq c \int_{I^m} \int_{-1}^1 \left| \sum_{j=0}^n c_{j,n}^\delta \sum_{k=0}^j \frac{k + \mu + \frac{d-1}{2}}{\mu + \frac{d-1}{2}} C_k^{(\mu + \frac{d-1}{2})}(u) \right. \\ & \quad \times \left. \left[\mathbb{P}_{j-k}^{(\alpha, \beta)}(\mathbf{x}) \right]^T \mathbb{P}_{j-k}^{(\alpha, \beta)}(\mathbf{e}) \right| (1-u^2)^{\frac{d-2}{2} + \mu} du w^{(\alpha, \beta)}(\mathbf{x}) d\mathbf{x}. \end{aligned}$$

After substituting in the well known identity for Gegenbauer polynomials,

$$\frac{n + \lambda}{\lambda} C_n^\lambda(x) = \tilde{C}_n^\lambda(1) \tilde{C}_n^\lambda(x),$$

we arrive at the following inequality.

$$\begin{aligned} & \int_{I^m} \int_{B^d} |K_n^\delta(\mathbf{x}, \mathbf{e}, \mathbf{y}, \mathbf{y}')| w(\alpha, \beta, \mu; \mathbf{x}, \mathbf{y}) \\ & \leq c \int_{I^m} \int_{-1}^1 \left| \sum_{j=0}^n c_{j,n}^\delta \sum_{k=0}^j \tilde{C}_k^{(\mu + \frac{d-1}{2})}(u) \tilde{C}_k^{(\mu + \frac{d-1}{2})}(1) \right. \\ & \quad \times \left. \left[\mathbb{P}_{j-k}^{(\alpha, \beta)}(\mathbf{x}) \right]^T \mathbb{P}_{j-k}^{(\alpha, \beta)}(\mathbf{e}) \right| (1-u^2)^{\frac{d-2}{2} + \mu} du w^{(\alpha, \beta)}(\mathbf{x}) d\mathbf{x}. \end{aligned}$$

Since the Gegenbauer polynomials are a subset of the Jacobi polynomials, the proof of our theorem then follows from Theorem 1.1 and Lemma 2.2 from [6], which, when combined, can be restated as follows.

Theorem 3.5. *Let $\alpha_i > -1$, $\beta_i > -1$, and $\alpha_i + \beta_i \geq -1$ for $1 \leq i \leq m$. If*

$$\delta > \sum_{i=1}^m \max\{\alpha_i, \beta_i\} + \frac{m}{2} + \max\left\{0, -\sum_{i=1}^m \{\alpha_i, \beta_i\} - \frac{m+2}{2}\right\},$$

then the integral

$$\int_{I^m} \left| \sum_{j=0}^n c_{j,n}^\delta \left[\mathbb{P}_j^{(\alpha, \beta)}(\mathbf{x}) \right]^T \mathbb{P}_j^{(\alpha, \beta)}(\mathbf{e}) \right| w^{(\alpha, \beta)}(\mathbf{x}) d\mathbf{x}.$$

is uniformly bounded for all n and $\mathbf{x} \in I^m$, which is sufficient to show that the Cesàro means of the Fourier orthogonal expansion of a continuous function f on I^m with respect to $w^{(\alpha, \beta)}$ converge in the norms of $L^p(I^m; w^{(\alpha, \beta)})$ for $1 \leq p < \infty$ and $C(I^m)$.

\square

REFERENCES

- [1] W. ZU CASTELL, F. FILBIR, Y. XU, Cesàro means of Jacobi expansions on the parabolic biangle, *J. Approximation Theory*, **159** (2009), 167-179.
- [2] FENG DAI, YUAN XU, Boundedness of projection operators and Cesàro means in weighted L^p space on the unit sphere, *Trans. Amer. Math. Soc.*, **361** (2009), 3189-3221.
- [3] STEFANO DE MARCHI, MARTINA MARCHIORO, ALVISE SOMMARIVA, Polynomial approximation and cubature at approximate Fekete and Leja points of the cylinder, arXiv:1110.5513v (2011).
- [4] CHARLES F. DUNKL, YUAN XU, *Orthogonal Polynomials of Several Variables*, Encyclopedia of Mathematics and its Applications, vol. **81**, Cambridge University Press, 2001.
- [5] GEORGE GASPER, Banach Algebras for Jacobi Series and Positivity of a Kernel, *The Annals of Mathematics*, **95** (1972), 261-280.
- [6] ZHONGKAI LI, YUAN XU, Summability of Product Jacobi Expansions, *Journal of Approximation Theory*, **104** (2000), 287-301.
- [7] CLAUS MÜLLER, *Spherical Harmonics*, Lecture Notes in Mathematics, vol. **17**, Springer-Verlag, 1966.
- [8] GABOR SZEGÖ, *Orthogonal Polynomials*, Colloquium Publications, vol. **23**, American Mathematical Society, 2000.
- [9] JEREMY WADE, A discretized Fourier orthogonal expansion in orthogonal polynomials on a cylinder, *J. Approximation Theory*, **162** (2010), 1545-1576.
- [10] YUAN XU, Summability of Fourier Orthogonal Series for Jacobi Weight on a Ball in \mathbb{R}^d , *Transactions of the American Mathematical Society*, **351** (1999), 2439-2458.

DEPARTMENT OF MATHEMATICS, PITTSBURG STATE UNIVERSITY, PITTSBURG, KS 66762.

E-mail address: `jwade@pittstate.edu`